Some model theory of flat modules

Flat modules can be hard to motivate, or even given an intuition on. The easiest motivation is, probably, that a flat module is a module that behaves "as simply as possible" under the tensor functor. Geometrically, a flat module might or might not correspond to the intuitive notion of a "family of algebraic varieties varying smoothly" — but this is, as every intuition, strongly dependent on the reader's own mental image and background. From a more algebraic point of view, 1.4 essentially states that any R-linear relation holding in a flat R-module M can be reconstructed from R-linear relations holding in in R. This means that, under flatness, any information held by M can be essentially recovered from information already known in R. If we only look at finitely generated modules, the difference between flat and projective modules becomes very subtle, and "information-wise", a projective module is morally a projection of a free module, which once again hints at the fact that all the information was already stored in R. The spirit of this short talk is exploring this idea of "information" held by a module from another point of view, that of first order logic. The first, natural question when scrutinizing a mathematical object — in this case, a flat module — with a logician's eye is whether or not the property that makes it so important is "first order", i.e. it can be expressed with a first order formula. In other words, can we write down a first order formula (or perhaps an infinite set of formulae) that completely characterize flat modules? The answer is surprising: this can be done if and only if the base ring R satisfies a certain condition, called *coherence*, which is purely algebraic in nature (i.e., it wasn't introduced by logicians for this specific task). One could arrive at coherence from another road, namely exploring the properties of the class of flat modules; in particular, this class is closed under direct products if and only if the ring is coherent. This establishes a profound connection between the algebraic and the model-theoretic world. The material exposed in this talk comes from [SE71]. A few of the results (and notably one implication of the main theorem, 1.6) will be black-boxed because of time and background reasons, but can be found in the references given below.

CAVEAT LECTOR:

Throughout this talk, by R we will denote a commutative ring with unity 1. The whole theory can be — and has been — developed without the assumption of commutativity, but we chose to avoid that for the sake of clarity.

1 Flat modules

The tensor product $\otimes_R$ defines, for every R-module N, a functor

$$- \otimes_R N : \underline{\mathrm{Mod}_R} \to \underline{\mathrm{Mod}_R}.$$

THEOREM 1.1

Let $M' \to M \to M'' \to 0$ be a short exact sequence, then

$$M' \otimes_R N \to M \to M'' \otimes_R N \to 0$$

is also short exact.

It is not true, in general, that short exact sequences of the form

$$0 \to M' \to M \to M'' \to 0$$

stay short exact when tensored.

EXAMPLE 1.2

A classical example is $\frac{\mathbb{Z}}{n\mathbb{Z}}$ as a $\mathbb{Z}$-module:

$$0 \to \mathbb{Z} \xrightarrow{f: x \mapsto nx} \mathbb{Z}$$

is exact, but

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{f \otimes_{\mathbb{Z}} \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}}$$

is not exact, since $\mathbb{Z} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}}{n\mathbb{Z}}$ and, under this isomorphism, $f \otimes \mathrm{id}$ becomes $g(x) = nx$, which is the zero map.

DEFINITION 1.3

An R-module N is said to be *flat* if the functor $- \otimes_R N$ is exact, i.e. if for every R-modules M, M', M'',

$$0 \to M' \to M \to M'' \to 0$$

is short exact if and only if

$$0 \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is short exact.

1.1 FLAT MODULES: REVISITED

Most of the material in this section comes from [Fai73]. The following theorem will be taken as a "black box", due to the sheer amount of theory that would be needed to get to its proof.

THEOREM 1.4 (*black box*)

Let $0 \to K \to F \to A \to 0$ be a short exact sequence of R-modules, where F is free. Then the following are equivalent,

1. A is flat,
2. for any $u \in K$ there is a morphism $\;: F \to K$ such that $\;(u) = u$,
3. for any $u_1, \dots u_n \in K$ there is a morphism $\;: F \to K$ such that $\;(u_i) = u_i$ for all $i = 1, \dots n$.

This allows us to rephrase flatness in terms of solutions to equations, achieving two goals: first, this makes flatness a notion more eloquently about *information*, since solutions to R-linear equations can be traced back to solutions to equations holding in R; second, this makes logicians slightly happier, because equations can be easily understood in the framework first order logic, while short exact sequences... not so much.

THEOREM 1.5

Let M be an R-module. Then the following are equivalent:

1. M is flat,
2. for all $a_1, \dots a_k \in M$ and $\;_1, \dots \;_k \in R$ such that $\;_1 a_1 + \dots + \;_k a_k = 0$, there exist $b_1, \dots b_l \in M$ and $\;_{ij} \in R$, for $i = 1, \dots l$ and $j = 1, \dots k$ such that every a_i decomposes as
$$a_i = \sum_{h=1}^{l} \;_{ih} b_h$$
and $\;_{i1} \;_1 + \dots + \;_{k1} \;_k = 0$ for all $i = 1, \dots l$,
3. for all $a_1, \dots a_k \in M$ and $\;_{ij} \in R$, $i \leq k$ and $j = 1, \dots s$, such that $\;_{1j} a_1 + \dots + \;_{kj} a_k = 0$ for all $j \leq s$, there exist $b_1, \dots b_l \in M$ and $\;_{ij} \in R$ for $i \leq l$ and $j \leq k$ such that every a_i decomposes as
$$a_i = \sum_{h=1}^{l} \;_{ih} b_i$$
and for each i, j we have $\sum_{h=1}^{k} \;_{kj} \;_{ik} = 0$.

Proof of 1.5: $1 \implies 3$: let $f : F \to M$ be a surjective morphism of R-modules, where F is a free R-module. Let $K = \ker(f)$, hence we get the short exact sequence
$$0 \to K \to F \to M \to 0.$$
By surjectivity of f, there are $x_1, \dots x_k \in F$ such that $f(x_i) = a_i$ for all $i \leq k$. Define
$$u_j = \sum_{h=1}^{k} \;_{hj} x_h$$
and notice that $f(u_j) = 0$ for all $j \leq s$, hence every $u_j \in K$. By 1.4, there is a morphism of R-modules $\;: F \to K$ that is trivial on the u_j's. Consider $x_j - \;(x_j)$:

since F is free, we can decompose

$$x_j - \quad (x_j) = \sum_{i=1}^{\ell} \quad {}_{ij} \quad {}_i$$

for some basis ${}_i$ of F. Call $b_i = f(\quad {}_i)$: we need to show these ${}_{ij}$'s and b_i's do what we want them to do, so

$$a_j = f(x_j) = f(x_j - \quad (x_j))$$

since $\quad (x_j) \in K$,

$$a_j = f(x_j - \quad (x_j)) = f\left(\sum_{i=1}^{\ell} \quad {}_{ij} \quad {}_i\right)$$

which by R-linearity turns into

$$a_j = \sum_{i=1}^{\ell} \quad {}_{ij} f(\quad {}_i) = \sum_{i=1}^{\ell} \quad {}_{ij} b_i.$$

Finally, since $\quad (u_i) = u_i$,

$$0 = u_j - \quad (u_j) = \sum_{i=1}^{k} \quad {}_{ij} x_i - \sum_{i=1}^{k} \quad {}_{ij} \quad (x_i),$$

and by grouping and using the decomposition of $x_i - \quad (x_i)$ we get,

$$0 = \sum_{i=1}^{k} \quad {}_{ij} \sum_{h=1}^{\ell} \quad {}_{hi} \quad {}_h = \sum_{h=1}^{\ell} \left(\sum_{i=1}^{k} \quad {}_{ij} \quad {}_{hi}\right) \quad {}_h,$$

which — since the ${}_h$'s form a base — implies

$$0 = \sum_{i=1}^{k} \quad {}_{ij} \quad {}_{hi}$$

for all j, h, as requested.

3 $\implies$ 2: this follows from the fact that 3 is just a generalization of 2.

2 $\implies$ 1: once again, let $f : F \to M$ be a surjective morphism of R-modules, where F is free, and suppose F has ${}_1, \dots \quad {}_n$ as a basis, for some n. Let $K = \ker(f)$, and suppose $u \in K$. We can write $u = \quad {}_1 \quad {}_{i_1} + \cdots + \quad {}_\ell \quad {}_{i_\ell}$ for some ℓ, and define $a_j = f(\quad {}_{i_j})$. It follows that $0 = f(u) = \quad {}_1 a_1 + \cdots + \quad {}_\ell a_\ell$, hence we can apply 2 and produce $b_1, \dots b_m \in M$ and ${}_{ij} \in R$ for $i \le m$ and $j \le \ell$ such that

$$a_j = \sum_{i=1}^{m} \quad {}_{ij} b_i, \quad {}_1 \quad {}_{i1} + \cdots + \quad {}_\ell \quad {}_{i\ell}, \text{ and } i \le m.$$

By surjectivity, there are $v_1, \dots v_m \in F$ such that $f(v_j) = b_j$. Define $\quad : F \to F$ on the basis by

$$(\quad {}_{i_j}) = \quad {}_{i_j} - \sum_{i=1}^{m} \quad {}_{ij} v_i$$

whenever $j \le \ell$, and define it to be zero otherwise. This way,

$$f(\quad (\quad {}_{i_j})) = a_j - \sum_{i=1}^{m} \quad {}_{ij} b_i = 0$$

so that $(F) \subseteq K$ and hence $: F \to K$, and

$$(u) = u,$$

as requested by the second condition of 1.4. This implies M is flat. ■

1.2 BASIC PROPERTIES AND COHERENCE

The title is slightly misleading, in that we will only focus on one basic property: it is arguably not the most important one, but it is a striking example of the fact that flat R-modules are not as well-behaved as one would expect.

But first, some good news.

THEOREM 1.6

If $\{M_i : i \in I\}$ is a family of R-modules, $\prod_{i \in I} M_i$ is flat if and only if each M_i is flat.

Here comes the bad news: we cannot extend this theorem to direct products. One can show that the class of flat R-modules is closed under (infinite) direct products if and only if R enjoys a certain algebraic property called "coherence".

DEFINITION 1.7

An R-module M is called *coherent* if every finitely generated submodule $N \subseteq M$ is finitely presented, i.e. there is a short exact sequence

$$0 \to F \to R^\ell \to N \to 0$$

for some finitely generated F and some integer ℓ. A ring R is coherent if it is coherent as an R-module.

REMARK 1.8

Suppose N is finitely presented, then for every short exact sequence

$$0 \to F' \to R^\ell \to N \to 0$$

we automatically get that F' is finitely generated. This is non-trivial, and uses some results from [Swa59].

For an example of a non-coherent ring, see 2.3.

THEOREM 1.9

For a commutative ring R, the following are equivalent

1. every direct product of flat R-modules is flat,
2. every direct product of copies of R is flat as an R-module,
3. any finitely generated R-submodule of a free R-module is finitely pre-

sented,

4. R is coherent.

Proof: 1 $\implies$ 2: this follows from the first property of 1.6.

2 $\implies$ 3: let M be free, and $N \subseteq M$ be finitely generated. Upon moving to a submodule of M containing N, we can assume M is finitely generated as well, i.e. $M \cong R^\ell$ for some ℓ. Let $N = \langle z_1, \dots z_s \rangle$, and define $f : R^s \to N$ by $f(e_i) = z_i$: this is an epimorphism with kernel $K = \ker(f)$. We hence get a short exact sequence

$$0 \to K \to R^s \xrightarrow{f} N \to 0.$$

We need to show that K is finitely generated, so that N is finitely presented. Consider the R-module $M' = R^K$, which is flat by hypothesis. For any $\quad = a_1(\quad)e_1 + \dots + a_s(\quad)e_s \in K$, we get

$$0 = f(\quad) = a_1(\quad)z_1 + \dots + a_s(\quad)z_s$$

hence if we define $\underline{a_i} = \{a_i(\quad), \quad \in K\}$, $\sum_{k=1}^s \underline{a_i} z_{ki} = 0$ for all $i = 1, \dots s$. Since M' is flat, there exist $b_1, \dots b_m \in M'$ and $\quad_{ij} \in R$ such that $\underline{a_k} = \sum_{i=1}^m \quad_{ik} b_i$ and $\sum_{j=1}^m \quad_{jk} z_{ij} = 0$ for all i, j, k. Define

$$x_i = \sum_{k=1}^r \quad_{ik} e_k, \ i = 1, \dots m.$$

For every i, $f(x_i) = \sum_{k=1}^r \quad_{ik} z_k = 0$, so that $x_1, \dots x_m \in K$. For every $\quad \in K$, we have $a_k(\quad) = \sum_{i=1}^m b_i(\quad) \quad_{ik}$ for every k and hence

$$= \sum_{k=1}^s a_k(\quad) z_k = \sum_{k=1}^s \sum_{i=1}^m b_i(\quad) \quad_{ik} z_k = \sum_{i=1}^m b_i(\quad) x_i,$$

so K is generated by $x_1, \dots x_m$.

3 $\implies$ 4: 4 is just a specific case of 3.

4 $\implies$ 2: consider a family of copies of R indexed by some set I, i.e. $\{R^i : i \in I\}$. We want to show that $R = \prod_{i \in I} R^i$ is flat. Suppose $\quad_1 a_1 + \dots + \quad_k a_k = 0$ holds in R, where $a_j \in R$ and $\quad_j \in R$. Let $J = \langle \quad_1, \dots \quad_k \rangle \subseteq R$ and F be a free module of rank k, say $F = \langle x_1, \dots x_k \rangle$. Define $f : F \to J$ to be the (epi)morphism defined by $f(x_j) = \quad_j$ and $K = \ker(f)$. Since we can write the short exact sequence

$$0 \to K \to F \xrightarrow{f} J \to 0$$

and J is finitely presented, by the remark above we get that K is finitely generated, say $K = \langle z_1, \dots z_r \rangle$. Rewrite $z_i = \sum_{h=1}^k \quad_{hi} x_h$ and let $u(i) = a_1(i)x_1 + \dots + a_k(i)x_k$ for every $i \in I$. We get that

$$f(u(i)) = a_1(i) \quad_1 + \dots + a_k(i) \quad_k = 0,$$

for every $i \in I$, hence we can write $u(i) = \sum_{j=1}^{r} b_j(i) z_j$ for some $b_j(i) \in R$. In particular,

$$a_1(i)x_1 + \cdots + a_k(i)x_k = u(i) = \sum_{j=1}^{r} b_j(i)z_j = \sum_{h=1}^{k}\sum_{j=1}^{r} b_j(i) \quad _{hj}x_h.$$

Since $x_1, \ldots x_k$ is a basis for F, we get

$$a_h(i) = \sum_{j=1}^{r} b_j(i) \quad _{hj}$$

and so $a_h = \sum_{j=1}^{r} b_j \quad _{hj}$. We are almost done: now

$$0 = f(z_i) = \sum_{h=1}^{k} \quad _{hi} f(x_h) = \sum_{h=1}^{k} \quad _{hi} \quad _h.$$

By 1.5, this means R is flat.

$3 \implies 1$: this requires some further work on certain additive functors. It can be found in [Fai73]. ■

2 Elementary classes of modules

Before introducing the natural question we are interested in we need to settle some terminology. We shall consider the so-called **language of R-modules**, that is a language containing

one constant symbol, 0, to be read as the zero of our R-module,

an operation symbol, +, to be read as addition between elements of our R-module,

a family of function symbols $\ _r$, indexed by $r \in R$, to be read as multiplication by the scalar r.

Altogether, the set of symbols $\{0, +\} \cup \{\ _r \mid r \in R\}$ will be denoted by $\mathcal{L}_R$. Despite being slightly confusing, we will usually conflate r with $\ _r$ and write $\ _r(x)$ as rx. We will call $\mathcal{L}_R$-**formula** any formula in this language, i.e. any Boolean combination of R-linear equations and inequalities, where some variables might be bounded. For example,

$$\exists y_1 \ldots \exists y_n [(r_1 y_1 + \cdots + r_n y_n = 0) \wedge \neg(s_1 y_1 + \cdots + s_n y_n = 0)]$$

for some fixed $r_1, \ldots r_n, s_1, \ldots s_n \in R$ is saying that there exists a family of elements $y_1, \ldots y_n$ that satisfy the first equation, but not the second. We will say that an R-module M **satisfies** a certain formula $!$, in symbols $M \models !$, if said formula is true in the R-module.

DEFINITION 2.1

> Let $\mathcal{C}$ be a class of R-modules. We will say that $\mathcal{C}$ is **elementary** if there is a family $\{ !_i \mid i \in I\}$ of $\mathcal{L}_R$-formulae such that
>
> $$\mathcal{C} = \{M \in \underline{Mod}_R \mid M \vDash ! \ \ \forall i \in I\}.$$

In other words, the *shared* information of all the R-modules in $\mathcal{C}$ is coded by the family of $\mathcal{L}_R$-formulae in question. Elementarity allows to reduce problems about the common properties of a certain class of R-modules to combinatorial properties of $\mathcal{L}_R$-formulae (and the sets that they define). This is the first step in a model-theoretic treatment – the next one would be to further reduce the kind of $\mathcal{L}_R$-formulae needed to describe a certain class, since in general the elements of the family $\{ !_i \mid i \in I\}$, also called *the axioms* of $\mathcal{C}$, can be arbitrarily complicated (for example, they might have an enormous – but finite – number of quantifiers). This procedure – "taming" the complexity of the formulae – is called *quantifier elimination*, and it has been proved for the theory of R-modules, but it is beyond the scope of this talk. We will consider the natural question: *when* is a certain class $\mathcal{C}$ of R-modules elementary? For example, the class of *all* R-modules is elementary, here are the axioms (see [ES71]):

$$\forall x \forall y \forall z[(x+y)+z = x+(y+z)],$$

$$\forall x[x+0 = x],$$

$$\forall x[x+(-1)\cdot x = 0],$$

$$\forall x \forall y[x+y = y+x],$$

$$\forall x[1\cdot x = x],$$

$$\forall x[(\quad + \quad)\cdot x = \quad \cdot x + \quad \cdot x], \ \forall \quad , \quad \in R,$$

$$\forall x \forall y[\quad \cdot (x+y) = \quad \cdot x + \quad \cdot y], \ \forall \quad \in R,$$

$$\forall x[\quad \cdot (\quad \cdot x) = (\qquad)\cdot x], \ \forall \quad , \quad \in R.$$

Notice that this set of formulae is infinite, as long as R is infinite. This is not an issue – most axiomatizations we know are actually infinite, with "axiom schematas" like the last three ones.

We now turn to flat R-modules. Looking at 1.4, one would expect the class of flat R-modules to be elementary. However, a more precise scrutiny of 1.4 shows that the b_i's extracted in condition 2 depend on the a_i's, hence we cannot write them *a priori* – logicians say "uniformly". Coherence allows, in some way, a "uniformization" of the solutions to these equations by turning finitely generated submodules into finitely presented submodules. In particular, the following theorem gives a surprising answer to our question.

THEOREM 2.2

> Let R be a ring and $\mathcal{C}$ be its class of flat R-modules. Then $\mathcal{C}$ is elementary if and only if R is coherent.

A few remarks: we will actually only show that if R is coherent, then its class of flat modules is elementary. The reverse implication requires some further model theory (particularly the use of ultrapowers, who would be too long to introduce in this short note).

EXAMPLE 2.3 (*See [Vak]*)

> Consider the ring of smooth functions on R and let $\mathcal{O}_p$ be its localization at the maximal ideal $\mathfrak{m}_p$, where $p = 0 \in \mathsf{R}$. It has a maximal ideal, again called $\mathfrak{m}_p$, generated by the class of t, i.e. $\mathfrak{m}_p = (t_p)$. Consider $\mathcal{O}_p$ as a module over itself, and let $!(t) = \text{"}_{(0,+\infty)}(t)\exp(-\frac{1}{t^2})$. Consider its equivalence class $!_p$ and the morphism $\mathcal{O}_p \to \mathcal{O}_p$ defined by $f_p \mapsto (f \cdot !)_p$. Its kernel, $I_!$, is the ideal of (germs of) functions that are zero for positive values of t. We have a short exact sequence
>
> $$0 \to I_! \to \mathcal{O}_p \to \mathrm{im}(\mathcal{O}_p) \to 0 :$$
>
> if $\mathcal{O}_p$ were coherent, then $\mathrm{im}(\mathcal{O}_p) \subseteq \mathcal{O}_p$ would be finitely presented and hence $I_!$ would be finitely generated. On the other hand, since $\mathfrak{m}_p$ is the unique maximal ideal it coincides with the Jacobson radical of $\mathcal{O}_p$ and hence $t_p \in J(\mathcal{O}_p)$. We have $t_p I_! = I_!$, so by Nakayama's lemma we would have $I_! = 0$, which is false. This means $\mathcal{O}_p$ is not coherent.

We now turn to the proof of 2.2, or rather of the sufficient condition for elementarity.

Proof of the right-to-left implication: Assume R is coherent. For any natural number $\ell \in \mathbb{N}$ and any vector $\_ = (\ _1, \dots \ _\ell) \in R^\ell$, consider the R-module morphism

$$f_\_ : R^\ell \to R$$

given by $(v_1, \dots v_\ell) \mapsto \sum_{i=1}^{\ell} \ _i v_i$. By the definition of coherence, $\ker(f_\_)$ is finitely generated, for example $\ker(f_\_) = \langle \ _1, \dots \ _s \rangle$. We now consider a first-order formula $!_\_$ given by

$$\forall a_1 \dots \forall a_\ell \left[\sum_{j=1}^{\ell} \ _j a_j = 0 \longrightarrow \exists y_1 \dots \exists y_s \bigwedge_{h=1}^{\ell} \left(a_h = \sum_{i=1}^{s} \ _{ih} y_i \right) \right],$$

where $\ _i = (\ _{i1}, \dots \ _{i\ell})$. Consider the set of formulae $\Sigma = \bigcup_{\ell \in \mathbb{N}} \{ !_\_ \mid \_ \in R^\ell \}$. We claim that

$$\Sigma \text{ holds in a } R\text{-module } M \text{ if and only if } M \text{ is flat}.$$

One implication is immediate once one sees that, altogether, the formulae $!_\_$ mean the equivalent condition for flatness proven in 1.5, hence a module where Σ holds is necessarily flat. On the other hand, assume M is flat. For any $\_ \in R^\ell$

and $a_1, \dots a_\ell \in M$, assume ${}_1 a_1 + \cdots + {}_\ell a_\ell = 0$ holds. We need to find $y_1, \dots y_s$ such that $a_h = \sum_{i=1}^{s} {}_{ih} y_i$ for all $h = 1, \dots \ell$. The equivalent conditions shown in 1.5 give us $b_1, \dots b_m$'s that could, hypothetically, play the role of the y_i's, but nothing guarantees that the ${}_{ih}$'s will be precisely the ${}_{ih}$'s. However, since

$$\sum_{h=1}^{\ell} {}_h \; {}_{ih} = 0$$

we get that, for every i, $({}_{i1}, \dots {}_{i\ell}) \in \ker(f\_)$, so we can write

$$({}_{i1}, \dots {}_{i\ell}) = \sum_{j=1}^{s} {}_{ij} ({}_{j1}, \dots {}_{js})$$

and in particular ${}_{ih} = \sum_{j=1}^{s} {}_{ij} \; {}_{jh}$. Hence,

$$a_h = \sum_{j=1}^{m} {}_{jh} b_j = \sum_{j=1}^{m} \left(\sum_{i=1}^{s} {}_{ji} \; {}_{ih} \right) b_j = \sum_{i=1}^{s} {}_{ih} \left(\sum_{j=1}^{m} {}_{ji} b_j \right).$$

Define $y_i = \sum_{j=1}^{m} {}_{ji} b_j$: the y_i's now witness that $!\_$ holds in M, hence proving the other implication. ■

3 Further developments

One of the reasons for introducing flat modules is to provide a generalization of important module-theoretic properties. The following section will be necessarily hand-wavy, but will try to give a glimpse of what happens to other classes of R-modules. We will need a couple of definitions.

DEFINITION 3.1

An R-module N is said to be **projective** if every short exact sequence of the form

$$0 \to M \to M' \to N \to 0$$

is split, i.e. – among many equivalent definitions – $M' \cong N \oplus M$.

A ring R is said to be **perfect** if every flat R-module is a projective R-module.

As exemplified by the picture below, if we define the following classes

$\mathcal{C}_f$ flat R-modules,

$\mathcal{C}_p$ projective R-modules,

$\mathcal{C}_\ell$ free R-modules,

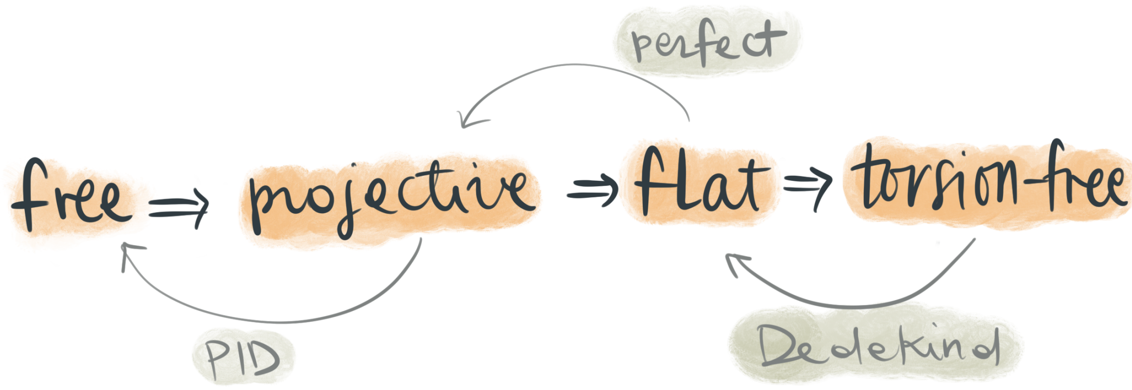

Figure 1: *Several generalizations of the notion of free modules, together with properties of the base ring R that provide the reverse implications.*

then we get

$$\mathcal{C}_\ell \subseteq \mathcal{C}_p \subseteq \mathcal{C}_f.$$

Under special assumptions on the ring R, some of this inclusions can be reversed (for example, if the ring is a PID we have $\mathcal{C}_p = \mathcal{C}_\ell$). We can then specialize — or generalize, depending on one's opinion — 2.2 to the following two theorems.

THEOREM 3.2

Consider a ring R. Then $\mathcal{C}_p$ is elementary if and only if R is coherent and perfect.

THEOREM 3.3

Consider a ring R. Then $\mathcal{C}_\ell$ is elementary if and only if either (1) R is artinian and local, or (2) R is artinian, finite and $R/J(R)$ is simple.

These two proofs actually build on the proof of 2.2. In particular, in 3.2 if R is perfect and coherent then $\mathcal{C}_p = \mathcal{C}_f$, hence it is elementary. Similarly, in 3.3 if R is artinian and local, it is also perfect and coherent, hence $\mathcal{C}_\ell = \mathcal{C}_f$; if R is finite and $R/J(R)$ is simple, then it is perfect and coherent, so $\mathcal{C}_f$ is elementary and with some more work we can write an explicit axiomatization of $\mathcal{C}_\ell$. This shows that, under algebraic hypotheses, other classes of R-modules can be fruitfully scrutinized with the tools of first order logic.

3.1 QUANTIFIER ELIMINATION

Finally, a few words on quantifier elimination. As explained in the introduction, while elementarity lays the groundwork for attacking module-theoretic problems with model-theoretic tools, something more has to be achieved in order to comfortably study a theory. This something is usually referred to as "quantifier elimination", and it boils down to the possibility of considerably sizing down the set of formulae that has to be analyzed in order to infer properties of the theory. Without quantifier elimination, one could have formulae with arbitrarily long sequences of quantifiers, something that makes proofs cumbersome and human understanding wishful thinking at best. When working on a new class of structures, then, the first tentatives are usually devoted to showing that every sentence (i.e., a formula where all variables are quantified) that can be written in the language is equivalent – in some precise sense – to a quantifier-free one. Quantifier-free formulae are evidently much simpler to understand than generic formulae: for example, in the language $\mathcal{L}_R$ quantifier-free formulae are conjunctions and disjunctions of R-linear equations and inequalities. In [Pre88], Chapter 16, Prest shows that complete quantifier elimination is actually equivalent to an algebraic notion on the base ring R, which is another proof of the pattern that kept showing up in the previous pages. Recall that a ring R is said to be **regular** if every R-module is flat. Then,

THEOREM 3.4

> A ring R is regular if and only if all R-modules have quantifier elimination.

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